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Dynamic option pricing with endogenous stochastic arbitrage

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ABSTRACT

Only few efforts have been made in order to relax one of the key assumptions of the Black–Scholes model: the no-arbitrage assumption. This is despite the fact that arbitrage processes usually exist in the real world, even though they tend to be short-lived. The purpose of this paper is to develop an option pricing model with endogenous stochastic arbitrage, capable of modelling in a general fashion any future and underlying asset that deviate itself from its market equilibrium. Thus, this investigation calibrates empirically the arbitrage on the futures on the S&P 500 index using transaction data from September 1997 to June 2009, from here a specific type of arbitrage called "arbitrage bubble", based on a t-step function, is identified and hence used in our model. The theoretical results obtained for Binary and European call options, for this kind of arbitrage, show that an investment strategy that takes advantage of the identified arbitrage possibility can be defined, whenever it is possible to anticipate in relative terms the amplitude and timespan of the process. Finally, the new trajectory of the stock price is analytically estimated for a specific case of arbitrage and some numerical illustrations are developed. We find that the consequences of a finite and small endogenous arbitrage not only change the trajectory of the asset price during the period when it started, but also after the arbitrage bubble has already gone. In this context, our model will allow us to calibrate the B–S model to that new trajectory even when the arbitrage already started.

1. Introduction

For almost 35 years, since the seminal articles by Black and Scholes [1] and Merton [2], the Black–Scholes (**B–S**) model has been widely used in financial engineering to model the price of a derivative on equity.[1] Thus, the research agenda in the option price modelling has been concentrated almost exclusively on testing the empirical validity of the model and relaxing some of the most restrictive assumptions of the original **B–S** model.

Indeed, the **B–S** model for an equity makes several well-known assumptions, such as: (i) the price of the underlying instrument follows a geometric Brownian motion, with constant drift μ and volatility σ (a lognormal random walk), (ii) it is possible to short sell the underlying stock, (iii) there are no dividends on the underlying, (iv) trading in the stock is continuous, in other words delta hedging is done continuously, (v) there are no transaction costs or taxes, (vi) all securities are perfectly divisible (it is possible to buy any fraction of a share), (vii) it is possible to borrow and lend at a constant risk-free interest rate r, (viii) there are no-arbitrage opportunities.

In analytic terms, if $B(t)$ and $S(t)$ are the risk-free asset and underlying stock prices, the price dynamics of the bond and the stock in this model are given by the following equations:

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1 Rubinstein [3] states that the **B–S** model is one of the most widely used formula in human history.

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$$\begin{aligned} dB(t) &= rB(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \end{aligned} \tag{1}$$

where r, μ and σ are constants and $W(t)$ is a Wiener process.

In order to price the financial derivative, it is assumed that it can be traded, so we can form a portfolio based on the derivative and the underlying stock (no bonds are included). Considering only non-dividend paying assets and no consumption portfolios, the purchase of a new portfolio must be financed only by selling from the current portfolio.

Calling $\vec{h}(t) = (h_S, h_\Pi)$ the portfolio, $\vec{P}(t) = (S, \Pi)$ the price vector of shares and $V(t)$ the value of the portfolio at time t; the dynamic of a self-financing portfolio with no consumption is given by

$$dV(t) = \vec{h}(t)d\vec{P}(t). \tag{2}$$

In other words, in a model without exogenous incomes or withdrawals, any change of value is due to changes in asset prices.

Another important assumption for deriving **B–S** equation is that the market is efficient in the sense that is free of arbitrage possibilities. This is equivalent with the fact that there exists a self-financed portfolio with value process $V(t)$ satisfying the dynamic:

$$dV(t) = rV(t)dt \tag{3}$$

which means that any locally riskless portfolio has the same rate of return than the bond.

For the classical model presented above, there exists a well-known solution for the price process of the derivative $\Pi(t)$ (See, for example Ref. [4]). Given its simplicity and predictive power this formulation can be described as one of the most popular standards in the profession.

Today however, it is possible to find models that have addressed and relaxed almost all of the assumptions mentioned above, including models with transaction costs, different probability distribution functions, stochastic volatility, imperfect information, etc; all of which have improved the prediction capabilities of the **B–S** model. See [4–7] for some complete reviews of these extensions.

Nevertheless, only few efforts have been made in order to address one of the key assumption of the model: the no-arbitrage assumption. Since the 80's economists have realized that, in a real market, futures contracts are not always traded at the price predicted by the simple no-arbitrage relation. Strong empirical evidence have supported this point many times and in different settings, however, economists have tended to develop several alternative explanations for the variability of the arbitrage, such as: differential tax treatment for spots and futures [8], and marking-to-market requirements for futures, but it was argued that noise is the main source of the mispricing [9]. It was also noted that there are certain factors that influence the arbitrage strategies and slow down the market reaction on the arbitrage. The factors include constrained capital requirements [10], position limits, and transaction costs [11]. Nevertheless, there have not been any attempts to modified the **B–S** model in terms of arbitrage processes, all modifications have been looking for alternative ways of facing the problem, for recent developments in this direction, see Refs. [12–14]. Notoriously, most of the attempts to incorporated explicitly arbitrage have been made by physicists. Perhaps, the reasons are the importance that academic economists give to the notion of equilibrium, or the sophisticated mathematical treatment necessary for modelling the arbitrage. Despite of this, it is clear that in practice there are arbitrage opportunities in the market, where a lot of people can make (lose) considerable sums of money everyday, see for example Refs. [15–17] among others.

Most of the attempts to take into account arbitrage in option pricing assume that the return from the **B–S** portfolio is not equal to the constant risk-free interest rate, changing the no-arbitrage principle (3) to an equation

$$dV(t) = (r + x(t))V(t)dt, \tag{4}$$

where $x(t)$ is a random arbitrage return. This formulation gives great flexibility to the model, since $x(t)$ can be seen as any deviations of the traditional assumed equilibrium, and not just as an arbitrage return. For instance, Ilinski [18] and Ilinski and Stepanenko [19] assume that $x(t)$ follows an Ornstein–Uhlenbeck process.

Other distinct effort in this direction is Otto [20], who reformulated the original **B–S** model through a stochastic interest rate. However, as Panayides [21] pointed out, the main problem with this approach is that the random interest rate is not a tradable security, and therefore the classical hedging cannot be applied. This difficulty leads to the appearance of an unknown parameter, the market price of risk, which cannot be directly estimated from financial data.

Finally, Panayides [21] and Fedotov and Panayides [22] follow an approach suggested by Papanicolaou and Sircar [23], where option pricing with stochastic volatility is modelled. These studies instead of finding the exact equation for option price focus on the pricing bands for options that account for random arbitrage opportunities. The approach yields pricing bands that are independent of the detailed statistical characteristics of the random arbitrage return.

The purpose of this paper is to develop an option pricing model with endogenous stochastic arbitrage, capable of modelling in a general fashion any underlying asset that deviate itself from its market equilibrium. To our knowledge this is the first attempt in this direction in the literature, modelling endogenously arbitrage process. It is expected that the introduction of this stylized fact of the financial markets could improve the forecasting performance of the original **B–S** model. It is important to notice that in the light of the recent events occurred in the global financial markets, where economists as well as financial analysts have been hardly questioned for their results and naive models, modelling efforts like this one, that assume more realistic views, considering departures from the traditional equilibrium approach, must be properly examined.[2] It is expected that given the flexibility of our approach, where arbitrage can vary in terms of shape, size and length, for example using "arbitrage bubbles", a new theoretical and empirical research agenda could be developed.

Section 2 of this paper describes the general model of option pricing with endogenous stochastic arbitrage, and it discussed its economic interpretation. Section 3 presents empirical evidence that supports our theoretical model and allow us to choose in a justified manner our main assumptions. Section 4 develops the analytic results of a particular function of the arbitrage amplitude, and some numerical results. Finally, conclusions and future research are developed.

2. The model

Deviation from the no-arbitrage assumption implies that investors can make profit in excess of the risk-free interest rate. For example, if $x(t)$ is greater than zero, then what we can do is: borrow from the bank, paying interest rate r, invest in the risk-free rate stock portfolio and make a profit. Alternatively, we could go short the option, delta hedging it.

Our setting assumes that arbitrage can be modelled using Eq. (4). Distinctively, we define $x(t)$ as a stochastic process identical to the Wiener process ($\mathrm{d}W$) used to model the trajectory of the underlying asset.

As it is well known, in a perfectly competitive market, assumed by the original **B–S** model, the action of buyers and sellers exploiting the arbitrage opportunity will cause the elimination of the arbitrage in the very short run.[3]

We will consider the **B–S** model in (1) and self-financing portfolio condition in (2). In what follows we state the following arbitrage condition:

$$\mathrm{d}V(t) = rV(t)\mathrm{d}t + f(t)V(t)\mathrm{d}W(t) \tag{5}$$

where $f(t) = f(t, S(t))$ is any function and W is the same Brownian motion in the dynamic of the underlying stock S.

Thus, we are assuming a model-dependent arbitrage, where the arbitrage possibilities are modelled with the same stochastic process that govern the underlying stock. This assumption allows us to linkage the arbitrage equation to the B–S original model.[4] This assumption is reasonable from a theoretical perspective for some kinds of arbitrages, which are inherent to the underlying asset, and endogenous in nature to the asset in analysis. The validity of this maintained hypothesis is tested empirically bellow, even though it is possible to find some clear evidence in this direction in the literature, see for instance Ref. [17].

Note that condition (5) can be rewritten as

$$\mathrm{d}V(t) = rV(t)\mathrm{d}t + f(t)V(t)\mathrm{d}W(t) = \left(r + f(t)\dot{W}(t)\right)V(t)\mathrm{d}t$$

where $\dot{W}$ is a white noise. This can be interpreted as a stochastic perturbation in the rate of return of the portfolio with amplitude f: $x(t) = f(t)\dot{W}(t)$.

2.1. General solution

We want to price the financial derivative under this arbitrage condition. We are going to derive its price dynamic as the solution $\Pi(t, S)$ of certain boundary value problem. In what follows we consider the price process depending on t, S, but we omit this dependence for the sake of simplicity.

Using Ito calculus we get:

$$\mathrm{d}\Pi = \frac{\partial \Pi}{\partial t}\mathrm{d}t + \frac{\partial \Pi}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\mathrm{d}S^2.$$

Given the dynamic for S in (1) we have:

$$\mathrm{d}\Pi = \left(\frac{\partial \Pi}{\partial t} + \mu S\frac{\partial \Pi}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 \Pi}{\partial S^2}\right)\mathrm{d}t + \sigma S\frac{\partial \Pi}{\partial S}\mathrm{d}W.$$

Self-financing portfolio condition in (2) can be understood as $\mathrm{d}V = h_S\mathrm{d}S + h_\Pi\mathrm{d}\Pi$. Considering this and (5) together and replacing dynamics for S and Π we get:

$$h_S\left(\mu S\mathrm{d}t + \sigma S\mathrm{d}W\right) + h_\Pi\left[\left(\frac{\partial \Pi}{\partial t} + \mu S\frac{\partial \Pi}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 \Pi}{\partial S^2}\right)\mathrm{d}t + \sigma S\frac{\partial \Pi}{\partial S}\mathrm{d}W\right]$$
$$= r\left(h_S S + h_\Pi \Pi\right)\mathrm{d}t + f\left(h_S S + h_\Pi \Pi\right)\mathrm{d}W.$$

2 [24] points out with respect to the financial crisis that: "the economic profession appears to have been unaware of the long build-up to the current worldwide financial crisis and to have significantly underestimated its dimensions once it started to unfold. In our view, this lack of understanding is due to a misallocation of research efforts in economics. We trace the deeper roots of this failure to the profession's insistence on constructing models that, by design, disregard the key elements driving outcomes in real-world markets".

3 In our setting we will considered implicitly the speed of market's adjustment by modelling an "arbitrage bubble", which can be defined in shape, duration and size, taking in this way into account the market clearance power.

4 Otherwise, the arbitrage should be modelled exogenously to the **B–S** model.

Collecting dt- and dW-terms we have:

$$h_S S(\mu - r) + h_\Pi \left(\frac{\partial \Pi}{\partial t} + \mu S \frac{\partial \Pi}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 \Pi}{\partial S^2} - r\Pi \right) = 0$$
$$h_S S(\sigma - f) + h_\Pi \left(\sigma S \frac{\partial \Pi}{\partial S} - f\Pi \right) = 0. \tag{6}$$

The condition for the existence of nontrivial portfolios (h_S, h_Π) satisfying (6) gives us the following:

Proposition 2.1. *Given the* **B–S** *model for a financial market in (1), self-financing portfolio condition (2) and stochastic arbitrage condition in (5) the price process Π of the derivative is the solution of the following boundary value problem in the domain* $[0, T] \times \mathbb{R}_+$.

$$\frac{\partial \Pi}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 \Pi}{\partial S^2} + \frac{r\sigma - \mu f}{\sigma - f} \left(S \frac{\partial \Pi}{\partial S} - \Pi \right) = 0$$
$$\Pi(T, s) = \Phi(s) \tag{7}$$

for constant r, μ, σ, any function f and a simple contingent claim Φ.

If we rescale the price axis taking $x = \ln s$, Eq. (7) changes to:

$$\frac{\partial \Pi}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \Pi}{\partial x^2} + \left(r^* - \frac{\sigma^2}{2} \right) \frac{\partial \Pi}{\partial x} - r^* \Pi = 0 \tag{8}$$

where $r^* = \frac{r\sigma - \mu f}{\sigma - f}$.

Thus, Eq. (8) shows a particular type of arbitrage, that occurs when the underlying asset and its arbitrage possibilities are generated by a common and endogenous stochastic process. This formulation is fairly general, in the sense that function f could take any functional form. In the next section, we are going to study how our model behave for a given amplitude function f.

2.2. Economic interpretation

We can notice that Eq. (7) is very similar to the **B–S** equation, except for the term r^*. In fact, when $f = 0$ or $\mu = r$, this equation is reduced to the **B–S** equation. Moreover, for any constant function $f \neq \sigma$, the solution of the boundary value problem (7) is again the **B–S** formula, but with the new rate r^*. In the arbitrage-free case, we know that the return on the stock is not allowed to dominate the return on the bond and vice versa, in this context the return of the bond is expected to be a convex combination of the higher and lower expected value of the return on the stock. Without arbitrage the rate of return of that portfolio must equal the short rate of interest.

In the case with arbitrage, if $f(t)\dot{W}(t) > 0$ then we can borrow money from the bank at the rate r and then invest it in the portfolio h_Π, where the money will grow at the rate $x(t)$. On the other hand, if $f(t)\dot{W}(t) < 0$ then we can sell the portfolio h_Π short and invest this money in the bank, and again there will be an arbitrage. Analogously, we can infer from Eq. (8) that when $r^* > r$ we will borrow money from the bank and invest it in the portfolio h_Π and, on the other hand, when $r^* < r$ we will short sell the underlying asset in order to invest in the risk-free rate. These two strategies in the presence of arbitrage will be studied below.

Fig. 1 describe jointly the relations between r^* and $\frac{f}{\sigma}$ in such a way that we can model different kind of scenarios, through an appropriate selection of parameter values.

Firstly, we can see from the first figure presented above that when $r < \mu$, for the case when $f < \sigma$, we will have that $r^* < r$ and hence we could short sell the underlying asset in order to invest in the risk-free rate and take advantage of the arbitrage possibility. On the other hand, when $f > \sigma$ we will have that $r^* > r$ and hence we could borrow money from the bank at the rate r and then invest it in the portfolio h_Π in order to make money from the arbitrage possibility. For the case when $r > \mu$ the reverse rule applies, so when $f > \sigma$ we will have that $r^* < r$ and when $f > \sigma$ we will have that $r^* < r$.

It is interesting to point out that when $f \to \infty$, in other words, the arbitrage is enormous, our model again become the **B–S** model with $r = \mu$, since $r^* = \mu$. This could easily be interpreted as that the arbitrage must be short and small, otherwise the market rapidly take advantage of the arbitrage possibility, that in this case is notorious.

This simple analysis could help us to define our investment strategy in the presence of arbitrage, depending upon the variables r, μ, σ and f and their relationships stated above. The first three variables are known and easy to estimate, while the arbitrage amplitude f must be defined in terms of the volatility of the underlying asset, in other words, whether it is expected that the arbitrage amplitude will be greater or smaller than σ.

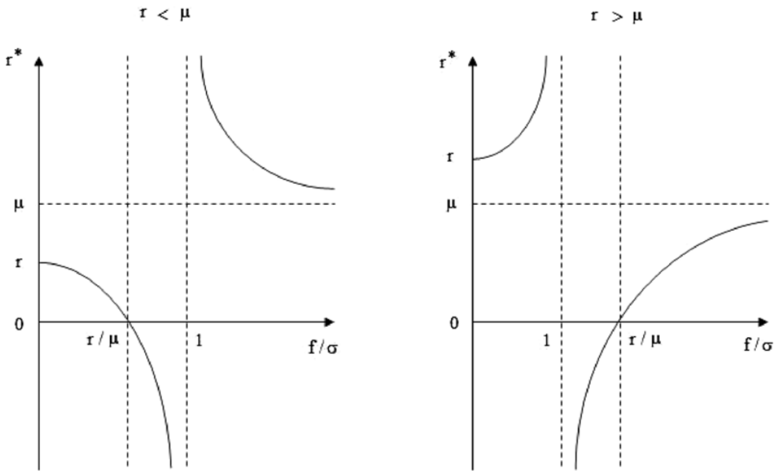

Fig. 1. Joint relations between r^* and $\frac{f}{\sigma}$.

3. Empirical evidence on arbitrage

In this section, we explore the presence of arbitrage on the futures of the S&P 500 index using transaction data from September 1997 to June 2009. Thus we use the same, but updated, data that Lo and MacKinlay [16], who are among the first economists to challenge the equilibrium view of the B–S model. In their, now classical work, they challenge the random walk assumption of the traditional B–S model. Since then, many studies have been carried out using their methodology, known today as the *Variance Test of Efficiency in Financial Markets*. This test of the presence of arbitrage can be carried out using the variance of the change of the log futures price divided by the variance of the change of the log index price. In the case of no-arbitrage market, this ratio has to be equal to 1 (since the contribution from the dividends and interest rates can be safely ignored at 15-minutes intervals). However, it turns out that the ratio differs considerably from 1 and the difference survives for intervals up to one trading day. As pointed out above several explanations have been elaborated to explain this market failure, among them the costs of trading. Nevertheless, as correctly pointed out by [17], these costs are considerably lower for the futures market, especially now considering the advances of electronic trading. This makes the futures market more suitable for speculations that bring higher volatility. Thus, the futures price is defined by both speculators and arbitrageurs, in contrast to the basic economic assumption of perfect markets. For [17], this leads to two possible conclusions: (i) one has to include speculators in pricing models; (ii) one might expect to find a correlation between the arbitrage return and the return on the underlying asset!

In computing the mispricing series, Lo and MacKinlay used quotes that were approximately 15 min apart, and each contract was followed from the expiration date of the previous contract until its expiration. Owing to the non-synchronous nature of the data, the error in the mispricing lifetime is of the order of 30 s and the typical time to take an arbitrage position could vary from 60 s to several minutes and even a trading day. For further details see [16]. It is important to mention that [17] in an also, very influential work, but this time for physicist, using the same data and methodology of Lo and MacKinlay, shows a very high correlation between the coefficients for the mispricing and the logarithm of the underlying asset.[5]

3.1. The experiment

Let us consider a forward contract in the frictionless market, with zero transaction cost and fixed interest rate r. Selling at time t a forward contract at forward price $F(t, T)$ with delivery time T. The selling party is obliged to sell at time T an agreed amount of the underlying asset at price $F(t, T)$.[6] It is straightforward to find the forward price from the no-arbitrage assumption. Consider the portfolio constructed at time t from the following:

1. The basket of stocks bought at the spot price $S(t)$ and held until the delivery date reinvesting the dividends continuously at the riskless rate till time T;
2. A debt of $S(t)$ dollars that were borrowed to finance acquisition;
3. The forward contract sold at the forward price $F(t, T)$.

5 We are very grateful to an anonymous referee for pointed this analysis out to us. Clearly this relationship would tend to support our main assumption, in which the spirit of this research is founded. In the rest of this section, we carried out an experiment that allow us to model a particular type of arbitrage. As Ilinsky, we follow closely Chapter 11 of Ref. [16].

6 For forward or futures contracts, there is no upfront payment. Thus, in contrast to futures, whose profits or losses are realized on an everyday marking-to-market basis, profits or losses on forward contracts are not realized until the delivery date.

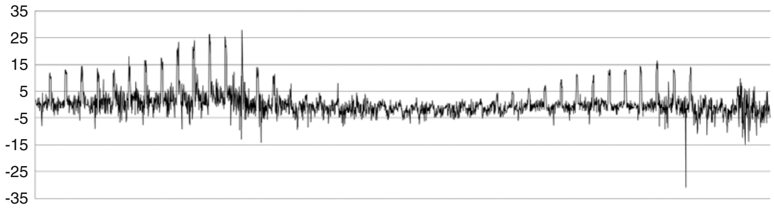

Fig. 2. Mispricing κ_0 of futures on S&P 500 index from September 1997 to June 2009.

The value of this portfolio at time t is equal to zero. At time T, the stocks are delivered and an amount $F(t, T)$ is received. To avoid certain losses or gains, this latter has to offset the debt accumulated on the bank account, which is equal to $S(t, T)e^{(r-d)(T-t)}$. Therefore, in the non-arbitrage world, the forward price is given by the equation:

$$F_0(t, T) = S(t, T)e^{(r-d)(T-t)}. \tag{9}$$

In the case where the forward price at the market, $F(t, T)$, is greater than the right-hand side of (9), a strategy that buys the index and sells the forward contract will earn riskless profit in excess of the risk-free rate r. This lifts the spot price of the basket and pushes the forward price down, thus moving prices closer to the no-arbitrage bound. If there is enough money available for the arbitrage, this strategy will lead to complete elimination of the arbitrage. A similar consideration holds for the opposite case where the forward contracts are cheaper than they have to be from (9). Since, we will be interested in the contracts on an index of stocks in this experiment, the underlying asset will represent a capitalization-weighted basket of stocks paying a continuous dividend rate d.

With deterministic interest rates, forward prices coincide with futures prices. As a result, we can use the same formula (9) to detect mispricing between the futures and the index.

As a quantitative measure of mispricing, we can use several quantities. The more obvious one is the *absolute mispricing* κ_0 which is the difference between the market futures price and its theoretical value given by (9):

$$\kappa_0(t, T) = F(t, T) - F_0(t, T). \tag{10}$$

In order to compare different assets or indexes and for technical reasons (that will be clarified below) we can use also as measures of arbitrage the so called *relative mispricing* κ_1 and *log-mispricing* κ_2 defined as:

$$\kappa_1(t, T) = \frac{F(t, T) - F_0(t, T)}{F(t, T)} \tag{11}$$

$$\kappa_2(t, T) = \ln F(t, T) - \ln F_0(t, T). \tag{12}$$

Note that the information about the random nature of the market is included, in different ways, in all the above magnitudes and that, since in a real market, futures contracts are not always traded at the price predicted by the simple no-arbitrage relation, all the above quantities tend to vary from 0.

3.2. Empirical results

We used daily financial data on the futures of the S&P 500 index from September 1997 to March 2009, as described below:

1. $F(t, T)$: Daily futures contracts price data on the S&P 500 index, obtained from *e-mini S&P 500 Complete History (ES)*. We considered the close figure of every day. The notional value of one contract is US$50 times the value of the S&P 500 stock index.
2. $S(t)$: Daily price of the underlying asset, in this case, the S&P 500 index close price adjusted for dividends and splits. We obtained this data directly from the site Yahoo Finance (http://finance.yahoo.com/q/hp?s=^GSPC). Specifically, from the variable *Adj Close* presented in all historical data.
3. r: The interest rate was obtained from the daily effective federal funds rate, which is a weighted average of rates on brokered trades, see http://www.federalreserve.gov/releases/h15/data.htm.

We can see from the following figure, the shape of the daily absolute mispricing κ_0 in the last 10 years, calculated according to Eq. (10). It is possible from the graph to see that in a long-term view there are some moments of greater mispricing and other of very low levels. In any case, the values are relatively low and some of them clearly are not profitable strategies (Fig. 2).

However, in a closer view we can see a clear pattern. Indeed, from the graphs of Fig. 3, we can see that expectation of the mispricing jumps a little before (5–6 days) the end of each contract and goes again to zero at the beginning of the next period. This behaviour is captured by all of the different measures of mispricing we defined.

At this point it is important to comment that any chosen functional form, with area different from zero and contained inside the mispricing found above, will perform better than the classical **B–S**.

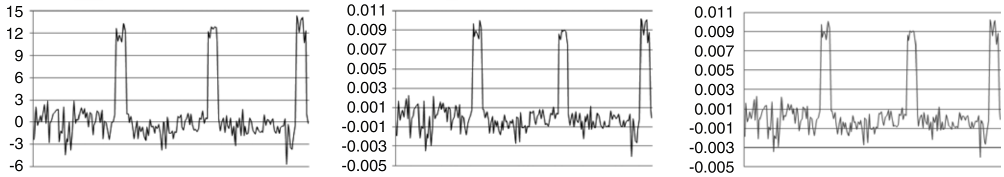

Fig. 3. Mispricing κ_0, κ_1 and κ_2 of futures on S&P 500 index from June 2006 to March 2007.

3.3. Defining the arbitrage functional form

Consider an underlying stock price process $S(t)$ with dynamics:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W(t) \tag{13}$$

where r, μ and σ are constants and $W(t)$ is a Brownian motion. The future spot price is given by

$$F_0(t) = G(S(t), t) = S(t)\mathrm{e}^{r(T-t)} \tag{14}$$

by applying Ito lemma we get the following stochastic equation for the future spot dynamics

$$\frac{\mathrm{d}F_0(t)}{F_0(t)} = (\mu - r)\mathrm{d}t + \sigma \mathrm{d}W(t) \tag{15}$$

so without arbitrage the future spot price F_0 and the underlying stock price S would must have the same volatility σ but different mean.

Actually the real price process $F(t)$ is not given by the spot price dynamics. In the spirit of [17], we postulate that the future price dynamics is given by:

$$\frac{\mathrm{d}F(t)}{F(t)} = (\mu - r)\mathrm{d}t + \sigma f(t)\mathrm{d}W(t) \tag{16}$$

where $f(t)$ is a deterministic time-dependent function. The explicit form of $f(t)$ has to be determined from the mispricing financial data showed above taking expectations to avoid randomness. Technical reasons determine the fact that the explicit form of function f does not appear when we use mispricing κ_0 for calculations. Instead of that we are going to show how to proceed for defining f in the cases of κ_1 and κ_2.

Given the mispricing

$$\kappa_1(t, T) = \frac{F(t, T) - F_0(t, T)}{F(t, T)}$$

applying Ito's Lemma we get

$$d\kappa_1 = \frac{F_0}{F}\left[\sigma(f - 1)\mathrm{d}W + \sigma^2(f - f^2)\mathrm{d}t\right]$$

where we drop the dependence on t and T, or equivalently

$$\kappa_1 = \int \frac{F_0}{F}\sigma^2(f - f^2)\mathrm{d}t + \int \frac{F_0}{F}\sigma(f - 1)\mathrm{d}W.$$

Taking expectation we obtain

$$g_1 \equiv E[\kappa_1] = \int \frac{F_0}{F}\sigma^2(f - f^2)\mathrm{d}t.$$

The time derivative of the above equation is

$$g_1'(t) = \frac{F_0(t)}{F(t)}\sigma^2(f(t) - f^2(t))$$

giving the explicit form for the function f:

$$f(t) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{F(t)}{F_0(t)}\frac{4g_1'(t)}{\sigma^2}}. \tag{17}$$

We call this particular function an "*arbitrage bubble*". The bubble wave shape is related to the time derivative $g_1'(t)$ of the mispricing expectation. Note that the condition for an arbitrage-free situation ($f = 1$) here is that $g_1'(t) \equiv 0$ which means that there is no mispricing in the period.

If we use the log-mispricing κ_2 and we reproduce the above calculations we get:

$$\kappa_2(t,T) = \ln F(t,T) - \ln F_0(t,T)$$
$$d\kappa_2 = \sigma(f-1)\mathrm{d}W + \frac{\sigma^2}{2}(1-f^2)\mathrm{d}t$$
$$g_2'(t) = \frac{\sigma^2}{2}(1-f^2)$$
$$f(t) = \sqrt{1-\frac{2g_2'(t)}{\sigma^2}}. \tag{18}$$

Again, the condition for no arbitrage is that $g_2'(t) \equiv 0$ giving no mispricing in the period. The use of κ_2 as a measure of mispricing, captures the jump behaviour of the signal as we saw and even more, gives a simpler way of determine the arbitrage bubble f given the financial data required. Note that this formulation allow us to determine any bubble shape of arbitrage given any series of financial mispricing.

For example, the trapezium like form of the expectation of log-mispricing κ_2 of Fig. 3 can be roughly estimated before the end of the contract by the function

$$g_2(t) = \begin{cases} 0 & t < T_1 \\ mt & T_1 < t < T_2 \\ h & T_2 < t < T \end{cases} \tag{19}$$

where $m = \frac{h}{T_2 - T_1}$ is the slope of linear growing part of the trapezium, h is its height, T_1 and T_2 the time instants where the mispricing appears, and T is the contract maturity time. The corresponding arbitrage bubble in this case is given by

$$f(t) = \begin{cases} 1 & t < T_1 \\ \sqrt{1-\frac{m}{\sigma^2}} & T_1 < t < T_2 \\ 1 & T_2 < t < T \end{cases} \tag{20}$$

which is a step-function-type bubble.

4. Modelling an arbitrage bubble in the Black–Scholes model

According to the last section step-function bubbles appear in real-world data so let us consider a t-step function f in $[0, T] \times \mathbb{R}$ of the form

$$f(t,x) = \begin{cases} C & [T_1, T_2] \times \mathbb{R} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

As mentioned above, we call this function f as "*arbitrage bubble*" in the interval $[T_1, T_2]$ of amplitude C. This bubble produces a constant stochastic arbitrage during a finite time interval and then vanishes.[7] Since, arbitrage possibilities are always short-lived, due to the market functioning, the arbitrage bubble is a very realistic function to model (Fig. 4).

As we discuss before, in any of the subintervals $[0, T_1]$, $[T_1, T_2]$ or $[T_2, T]$; the price process Π of the derivative follows a **B–S** formula, with constant rates r, r^*, and r again respectively, but the corresponding boundary value problems use different boundary conditions depending on the evolution of the price.

Recalling the usual **B–S** propagator P_{BS}

$$P_{BS}(T,r,t,x,x') = \mathrm{e}^{-r(T-t)} \frac{1}{\sqrt{2\pi(T-t)\sigma^2}} \mathrm{e}^{-\frac{1}{2(T-t)\sigma^2}\{x-x'+(T-t)(r-\sigma^2/2)\}^2} \tag{22}$$

we can construct the solution of (8) defining the following arbitrage propagator

$$P_{BSA}(t,x,x') = \begin{cases} P_{BS}(T_1, r, t, x, x') & \text{for } t \in [0, T_1] \\ P_{BS}(T_2, r^*, t, x, x') & \text{for } t \in [T_1, T_2] \\ P_{BS}(T, r, t, x, x') & \text{for } t \in [T_2, T]. \end{cases} \tag{23}$$

[7] It is important to remark, that this kind of functional form could be used in a recursive manner to model almost any case of arbitrage.

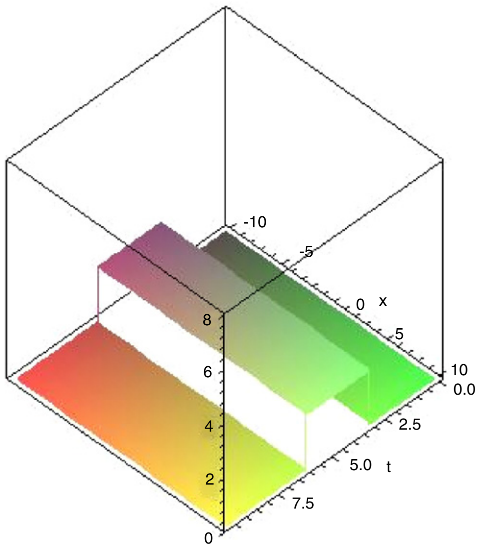

Fig. 4. A t-step function simulating an "arbitrage bubble".

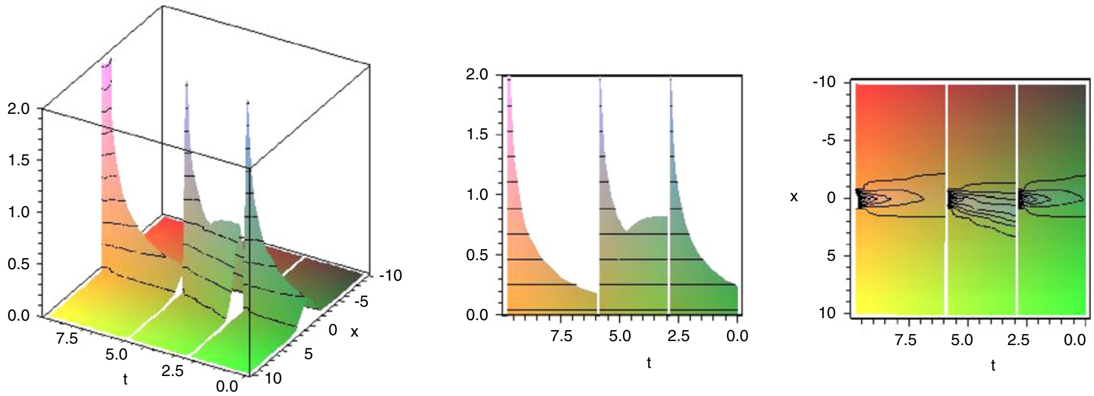

Fig. 5. Propagator for $r = 0.2$, $\mu = 0.6$, $\frac{f}{\sigma} = 0.5$.

Different sets of parameter values for r, μ, σ and f give different behaviours for the propagator during the arbitrage window. It can change or not the direction of propagation and can be amplitude-decreasing or increasing. As it was discussed above different relationships between r^* and $\frac{f}{\sigma}$ can model different kind of propagator's scenarios, through an appropriate selection of parameter values.

Note that these propagators consist on gaussians with moving maxima and which amplitude vary at exponential rate. At times T, T_1 and T_2 they reduce to Dirac's deltas. In other words, it is the model that claim for small and short arbitrage possibilities!

In Fig. 5, we can see that different values for r^* and $\frac{f}{\sigma}$ give different behaviours for the propagator during the arbitrage, changing amplitude and direction, and it could be increasing and decreasing through time. It is possible to notice as well that r^* could take a negative value.

In this case, the solution of the boundary condition problem (8) is

$$\Pi(t,x) = \int_{\mathbb{R}} P_{BSA}(x,t,x')\Psi(t,x')\mathrm{d}x' \tag{24}$$

where $\Psi(t,x')$ is the step border condition function

$$\Psi(t,x') = \begin{cases} \Pi(T_1,x') & \text{in } [0,T_1]\times\mathbb{R} \\ \Pi(T_2,x') & \text{in } [T_1,T_2]\times\mathbb{R} \\ \Phi(\mathrm{e}^{x'}) & \text{in } [T_2,T]\times\mathbb{R}. \end{cases} \tag{25}$$

We can compute explicit solutions of Eq. (8) given certain simple contingent claims as boundary conditions (Fig. 6).

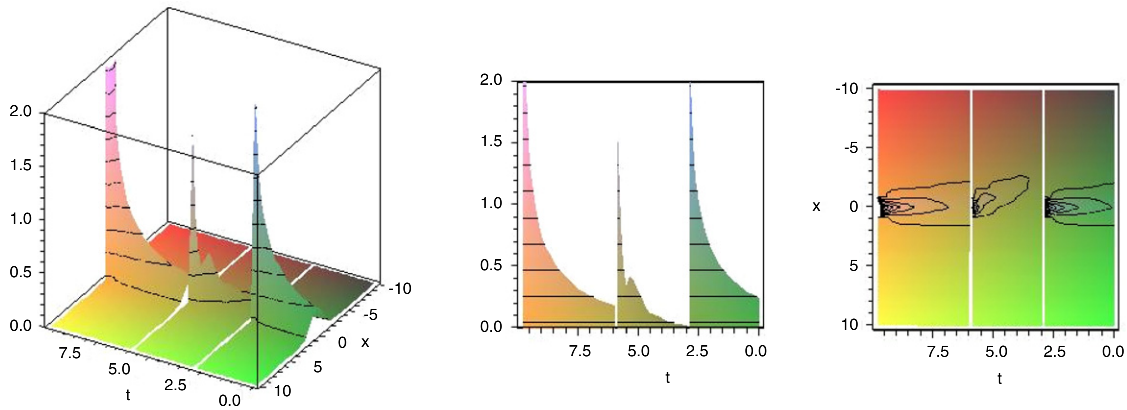

Fig. 6. Propagator for r $= 0.2$, $\mu = 0.6$, $\frac{f}{\sigma} = 2$.

4.1. Analytical results

The next results show the analytical solutions for Binary and European call options.

Proposition 4.1. *The price of a Binary call option with strike price K and time of maturity T under a **B–S** model with an endogenous arbitrage bubble as in* (21) *is given by the formula*

$$\Pi(t,x) = \begin{cases} \Pi_1(t,x) & si\ 0 \leqslant t \leqslant T_1 \\ \Pi_2(t,x) & si\ T_1 \leqslant t \leqslant T_2 \\ \Pi_3(t,x) & si\ T_2 \leqslant t \leqslant T \end{cases} \quad (26)$$

with

$$\Pi_1(t,x) = \frac{K\mathrm{e}^{-r(T_1-t)-r^*(T_2-T_1)-r(T-T_2)}}{2\pi}\int\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{u^2+z^2}{2}} N[d_2(T_2, d_3(T_1, d_4(t,x,z),u))]\mathrm{d}u\mathrm{d}z$$
$$\Pi_2(t,x) = \frac{K\mathrm{e}^{-r^*(T_2-t)-r(T-T_2)}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-u^2/2} N[d_2(T_2, d_3(T_1,x,u))]\mathrm{d}u$$
$$\Pi_3(t,x) = \mathrm{e}^{-r(T-t)}KN[d_2(t,x)].$$

Proposition 4.2. *The price of a European call option with strike price K and time of maturity T under a **B–S** model with an endogenous arbitrage bubble as in* (21) *is given by the formula*

$$\Pi(t,x) = \begin{cases} \Pi_1(t,x) & si\ 0 \le t < T_1 \\ \Pi_2(t,x) & si\ T_1 \le t < T_2 \\ \Pi_3(t,x) & si\ T_2 \le t < T \end{cases} \quad (27)$$

with

$$\Pi_1(t,x) = \frac{\mathrm{e}^{x-\frac{\sigma^2}{2}(T_2-t)}}{2\pi}\int\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{u^2+z^2}{2}-u\sigma\sqrt{T_2-T_1}-z\sigma\sqrt{T_1-t}} N[d_1(T_2, d_3(T_1, d_4(t,x,z),u))]\mathrm{d}u\mathrm{d}z$$
$$\quad - \frac{K\mathrm{e}^{-r(T_1-t)-r^*(T_2-T_1)-r(T-T_2)}}{2\pi}\int\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{u^2+z^2}{2}} N[d_2(T_2, d_3(T_1, d_4(t,x,z),u))]\mathrm{d}u\mathrm{d}z$$
$$\Pi_2(t,x) = \frac{\mathrm{e}^{x-\frac{\sigma^2}{2}(T_2-t)}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-u^2/2-u\sigma\sqrt{T_2-t}} N[d_1(T_2, d_3(T_1,x,u))]\mathrm{d}u$$
$$\quad - \frac{K\mathrm{e}^{-r^*(T_2-t)-r(T-T_2)}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-u^2/2} N[d_2(T_2, d_3(T_1,x,u))]\mathrm{d}u$$
$$\Pi_3(t,x) = \mathrm{e}^{x}N[d_1(t,x)] - \mathrm{e}^{-r(T-t)}KN[d_2(t,x)].$$

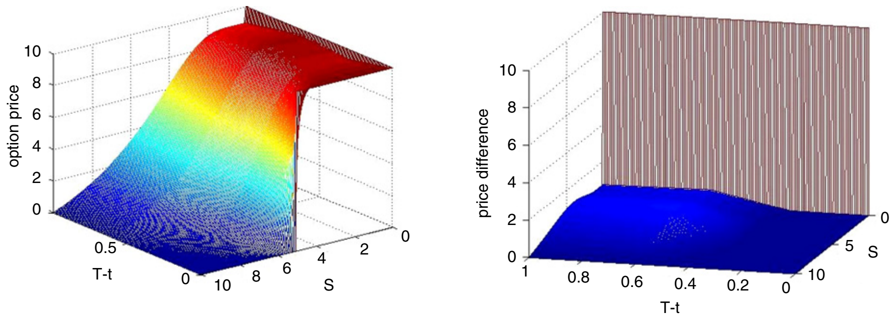

Fig. 7. Price of a Binary option and differences with B–S model for $\mu = 0.6$ and $f = 0.5 \cdot \sigma$.

In the propositions above N is the cumulative distribution function for the $N[0, 1]$ distribution, $x = \ln s$ and

$$d_1(t,x) = \frac{1}{\sigma\sqrt{T-t}}\left[x - \ln(K) + (r + \sigma^2/2)(T-t)\right]$$
$$d_2(t,x) = \frac{1}{\sigma\sqrt{T-t}}\left[x - \ln(K) + (r - \sigma^2/2)(T-t)\right]$$
$$d_3(t,x,u) = x - u\sigma\sqrt{T_2 - t} + (r^* - \sigma^2/2)(T_2 - t)$$
$$d_4(t,x,z) = x - z\sigma\sqrt{T_1 - t} + (r - \sigma^2/2)(T_1 - t).$$

We can see from both call options results showed above that:

1. $\Pi_3(t,x)$, $d_1(t,x)$ and $d_2(t,x)$ are identical to the **B–S** solutions.
2. $\Pi_2(t,x)$ and $\Pi_1(t,x)$ are calculated almost in a recursive form.
3. $d_3(t,x,u)$ and $d_4(t,x,z)$ include new parameters such as u, z, and $r*$.

In general, we can see from these results that the consequences of an endogenous arbitrage, not only change the trajectory of the asset price during the period when it started, but also after the arbitrage bubble is already gone. It is important to remark that this change in the price's trajectory will imply that if we are using the standard **B–S** model we will have to recalibrate it every time that an arbitrage bubble occurs, independently of how small or short-lived is the process. On the other hand, our model has the virtue of capture this new trajectory, bridging the possibility of self-calibration before an arbitrage possibility. Finally, we can add that this analytical formulation is not very different to that of the classical **B–S** model, with a very similar interpretation as well, which potentially gives future applicability to our formulation.

4.2. Some numerical illustrations

In this section a numerical modelling is developed for Eq. (7). This typical partial differential equation is solved using finite differences, specifically the Crank–Nicolson Method. A discrete equation for the endogenous arbitrage equation can be written as follows:

$$\frac{\Pi_i^{k+1} - \Pi_i^k}{\Delta t} + \frac{1}{2}\sigma^2 S^2\left[\frac{\Pi_{i-1}^{k+1} - 2\Pi_i^{k+1} + \Pi_{i+1}^{k+1} + \Pi_{i-1}^k - 2\Pi_i^k + \Pi_{i+1}^k}{2\Delta x^2}\right] + r^* S\left(\frac{\Pi_{i+1}^k - \Pi_{i-1}^k}{\Delta x}\right) - r^*\Pi_i^k = 0.$$

Two numerical models were programmed for a Binary and European options using MATLAB. The results are shown below. For both options two charts are presented.

Firstly, the complete trajectory of the underlying asset considering endogenous stochastic arbitrage, and secondly, the difference between the **B–S** model and our model with arbitrage. The latter chart could be interpreted as the contribution of the arbitrage to the price, which could be positive or negative, giving a numerical approximation of the deviation of the traditional free risk rate of the portfolio, or the equilibrium state. Indeed, as discussed above if the contribution of the arbitrage is positive (i.e. there is a increase in the price of the stock) and big enough to dominate the free risk asset, the trader should borrow money from the bank at the rate r and then invest it in the asset Π. In the specific examples showed here, we are modelling an arbitrage bubble equivalent to $f = 0.5 \cdot \sigma$ and other equal to $f = 2 \cdot \sigma$. In other words, firstly, we want to model the impact of an arbitrage with an amplitude comparable to half the standard deviation of the underlying asset's price. After that we model a twice standard deviation arbitrage process.

5. Conclusions and further research

To the best of our knowledge this is the first model that considers endogenous stochastic arbitrage. This formulation allow us to define an investment strategy in the presence of arbitrage, in a similar fashion to that of the classical **B–S** model.

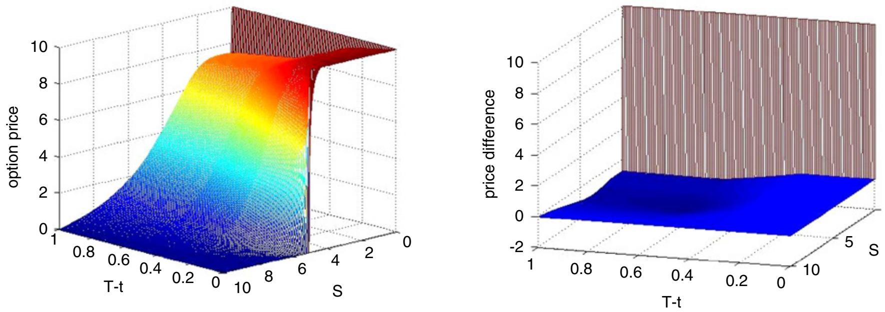

Fig. 8. Price of a Binary option and differences with B–S model for $\mu = -0.6$ and $f = 0.5 \cdot \sigma$.

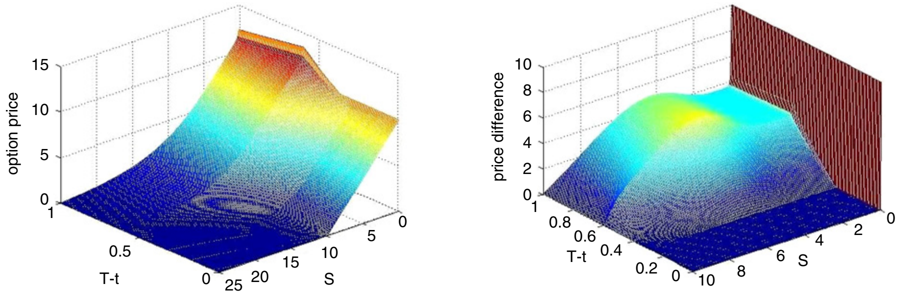

Fig. 9. Price of a European option and differences with B–S model for $\mu = 0.6$ and $f = 2 \cdot \sigma$.

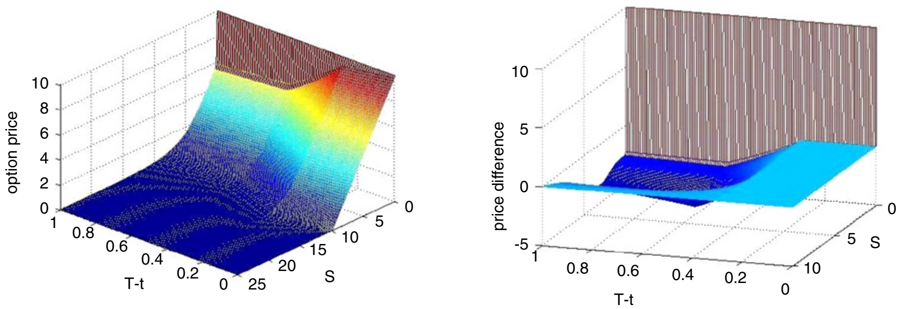

Fig. 10. Price of a European option and differences with B–S model for $\mu = -0.6$ and $f = 2 \cdot \sigma$.

Basically, if the trader could anticipated an arbitrage possibility (its amplitude) that is generated endogenously for that specific sector, he can predict whether or not one asset is going to dominate the another, or in other words, whether an arbitrage possibility could be detected.

Indeed, in terms of the first conclusions of our general model, we can point out that: if $f(t)\dot{W}(t) > 0$ then we can borrow money from the bank at the rate r and then invest it in the portfolio h_Π, where the money will grow at the rate $x(t)$. On the other hand, if $f(t)\dot{W}(t) < 0$ then we can sell the portfolio h_Π short and invest this money in the bank, and again there will be an arbitrage (Figs. 7–10).

Secondly, it is possible to model different arbitrage functions and find the analytical solutions of the model. We present the case of an *arbitrage bubble*, which can easily model a finite and short-lived arbitrage process. The selection on this type of arbitrage come out of an empirical investigation, using data from S&P 500 index from September 1997 to June 2009 in a similar fashion to Refs. [17,16]. In general, we can see from these results that the consequences of an endogenous arbitrage, not only change the trajectory of the asset price during the period when it started, but also after the arbitrage bubble is already gone. In this context, our model will allow us to calibrate the B–S model to that new trajectory even when the arbitrage already started.

Summing up, this paper extends the classical **B–S** model allowing for short arbitrage possibilities and sometime negative interest rate, which could not be modelled using **B–S**. Besides, all the results and intuition behind this new formulation rely only on the f parameter, which is a relative measure of the arbitrage's amplitude with respect to the model original σ.

Future research should be aim to empirically proof the model, evaluating in an ex-post manner the predictive power of its investment strategy recommendations. Since it is easy to find small arbitrage possibilities, defined as deviations of the traditional **B–S** model, from the real data, would be very useful to test this model for different assets. Complementary to this point, it would be interesting to try other different functional forms for modelling arbitrage episodes, some goods candidates are: decreasing exponential functions, linear function with negative slopes, and nested t-step function. Finally, it could be of much interest to include arbitrage possibilities to the **B–S** model with several underlying assets, since in many markets, like the commodities, short markets misalign are frequent.

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